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INSTABILITY OF AN ELASTIC MATERIAL

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Abstract—Conditions are derived which are necessary for stability of incompressible elastic materials. These are obtained by considering the speeds of small-amplitude plane waves superposed on a finitely deformed state of the material.

1. INTRODUCTION

IN PREVIOUS papers, Toupin and Bernstein [1] and Hayes and Rivlin [2] have discussed the propagation of plane small amplitude sinusoidal waves in an elastic material subjected to an initial finite static deformation. In [2] the secular equation was obtained for the wave speeds when the material is isotropic and the static deformation is pure homogeneous. Further, it was pointed out that if any of the wave speeds satisfying the secular equation are non-real, the material must be inherently unstable. In the present paper, we consider certain conditions on the strain-energy function which ensure this instability, in the case when the material is incompressible. Ericksen [3] has discussed the propagation of a second-order discontinuity in a deformed incompressible isotropic elastic material. He obtained conditions for the principal velocities of propagation for transverse second-order discontinuities to be real. Now, the velocities of propagation of a sinusoidal small-amplitude wave in an elastic material, subjected to a homogeneous static deformation, are the same as those for a propagating second-order discontinuity, with the same directions of polarization and propagation. It follows that if the conditions obtained by Ericksen in [3] are violated, the material is necessarily unstable.

In the present paper, we obtain further conditions which ensure material instability. These could undoubtedly be obtained by appropriate specialization of the formalism of Ericksen. However, we prefer to derive them directly from first principles, following a procedure analogous to that adopted in the compressible case by Hayes and Rivlin [2]. Accordingly, in Section 2 we consider a small-amplitude plane sinusoidal wave to be propagated in an arbitrary direction, in an incompressible isotropic elastic material subjected to an initial static deformation and obtain a formula for the incremental Cauchy stress associated with this deformation. Using this and the incremental equations of motion, we obtain the secular equation in the case when the wave is propagated in a direction lying in a principal plane and is linearly polarized in this plane. A condition is obtained which, if satisfied, ensures that all of the wave speeds for such waves are real and, if violated ensures that, for at least one direction in the principal plane, the wave speeds are imaginary and the material is therefore unstable.

2. SMALL DEFORMATION SUPERPOSED ON PURE HOMOGENEOUS DEFORMATION

We consider an isotropic, incompressible elastic material to be subjected to a static pure homogeneous deformation with principal extension ratios λ_1, λ_2, λ_3. We take as reference system a rectangular cartesian coordinate system x and consider that, in this pure homogeneous deformation, a particle initially at ξ_α moves to X_i. The pure homogeneous deformation is then described by

$$X_1 = \lambda_1\xi_1, \qquad X_2 = \lambda_2\xi_2, \qquad X_3 = \lambda_3\xi_3. \tag{2.1}$$

Now, suppose that superposed on this pure homogeneous deformation, we have a plane sinusoidal wave, whose wave normal is in the direction of the unit vector l_j and whose angular frequency is ω. The planes of constant phase and constant amplitude for the wave are assumed to be the same. We assume that the amplitude of the superposed wave is sufficiently small that we may neglect terms of second and higher degrees in the displacement components associated with it, in comparison with those of first degree. We may accordingly conveniently use the complex notation in describing this wave. Let u_i be the complex displacement vector associated with this wave. We write

$$u_i = U_i \exp[\iota\omega(Sl_iX_i - t)] \tag{2.2}$$

where S is the complex slowness, t denotes time, $\iota = \sqrt{-1}$ and U_i is a constant vector. We restrict our discussion to waves which are linearly polarized in a constant direction. In this case, we may, without loss of generality, take U_i to be real. This is equivalent to specifying the phase at $x_i = 0$, $t = 0$.

As a result of the deformations described by (2.1) and (2.2) a particle which was initially at ξ_i moves to x_i, where†

$$x_i = X_i + u_i^+. \tag{2.3}$$

Let σ_{ij} be the Cauchy stress associated with the resultant deformation (2.3). We may write

$$\sigma_{ij} = \Sigma_{ij} + \bar{\sigma}_{ij}^+, \tag{2.4}$$

where Σ_{ij} is the stress associated with the pure homogeneous deformation (2.1) and $\bar{\sigma}_{ij}$ is the complex incremental stress associated with the deformation (2.2).

For an isotropic incompressible elastic material, the strain energy W per unit volume is expressible as a function of the strain invariants i_1 and i_2 defined by

$$i_1 = \operatorname{tr}\mathbf{c}, \qquad i_2 = \tfrac{1}{2}\{(\operatorname{tr}\mathbf{c})^2 - \operatorname{tr}\mathbf{c}^2)\}, \tag{2.5}$$

where $\mathbf{c}$ is the Finger strain defined by†

$$\mathbf{c} = \|c_{ij}\| = \|x_{i,\alpha}x_{j,\alpha}\|. \tag{2.6}$$

The fact that all deformations of an incompressible material are necessarily isochoric may be expressed by the relation

$$i_3 = 1, \tag{2.7}$$

† Throughout the paper we shall use superscripts $^+$ and $^-$ to denote the real and imaginary parts respectively of a complex quantity.

‡ We use the notation $_{,\alpha}$ for $\partial/\partial\xi_\alpha$.

where i_3 is the strain invariant defined by

$$i_3 = \det \mathbf{c}. \qquad (2.8)$$

We may write

$$i_j = I_j + \bar{i}_j^+ \qquad (j = 1, 2, 3), \qquad (2.9)$$

where I_j is the value of i_j associated with the pure homogeneous deformation (2.1) and $\bar{i}_j$ are the complex incremental strain invariants due to the deformation (2.2). Also, c_{ij} may be expressed as

$$c_{ij} = C_{ij} + \bar{c}_{ij}^+, \qquad (2.10)$$

where C_{ij} is the Finger strain associated with the deformation (2.1) and $\bar{c}_{ij}$ is the complex incremental Finger strain due to the deformation (2.2). From (2.6) we have, with (2.1), (2.2) and (2.3),

$$C_{11} = \lambda_1^2, \qquad C_{22} = \lambda_2^2, \qquad C_{33} = \lambda_3^2, \qquad C_{ij} = 0 \qquad (i \neq j) \qquad (2.11)$$

and*

$$\bar{c}_{AB} = \iota\omega S(\lambda_A^2 U_B l_A + \lambda_B^2 U_A l_B)\exp[\iota\omega(Sl_iX_i - t)].$$

From (2.5) and (2.7)–(2.11), we obtain

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \qquad I_2 = \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 + \lambda_1^2\lambda_2^2, \qquad I_3 = \lambda_1^2\lambda_2^2\lambda_3^2 = 1 \qquad (2.12)$$

and

$$\begin{aligned} \bar{i}_1 &= 2\iota\omega S \sum_{A=1}^{3} \lambda_A^2 U_A l_A \exp[\iota\omega(Sl_iX_i - t)], \\ \bar{i}_2 &= 2\iota\omega S \sum_{A=1}^{3} \lambda_A^2(I_1 - \lambda_A^2) U_A l_A \exp[\iota\omega(Sl_iX_i - t)], \\ \bar{i}_3 &= 2\iota\omega S \sum_{A=1}^{3} U_A l_A \exp[\iota\omega(Sl_iX_i - t)] = 0. \end{aligned} \qquad (2.13)$$

For an isotropic incompressible elastic material, the Cauchy stress σ_{ij} may be expressed in terms of the strain energy per unit volume by the formulae

$$\sigma_{ij} = t_{ij} - p\delta_{ij}, \qquad (2.14)$$

where

$$\begin{aligned} t_{ij} &= 2[(w_1 + i_1 w_2)c_{ij} - w_2 c_{ik}c_{kj}], \\ w_1 &= \partial W/\partial i_1, \qquad w_2 = \partial W/\partial i_2, \end{aligned} \qquad (2.15)$$

and p is an arbitrary hydrostatic pressure. In accordance with equation (2.4), we can write

$$t_{ij} = T_{ij} + \bar{t}_{ij}^+, \qquad p = P + \bar{p}^+, \qquad (2.16)$$

* We use the notation $_{,A}$ for $\partial/\partial\xi_A$, but do not adopt the summation convention for capital subscripts.

where T_{ij} and P are the values of t_{ij} and p respectively for the deformation (2.1) and $\bar{t}_{ij}$ and $\bar{p}$ are their complex increments associated with the superposed deformation (2.2).

From (2.15), we can write similarly

$$w_1 = W_1 + \bar{w}_1^+, \qquad w_2 = W_2 + \bar{w}_2^+, \tag{2.17}$$

where W_1 and W_2 are the values of w_1 and w_2 for the deformation (2.1) and $\bar{w}_1$ and $\bar{w}_2$ are their complex increments due to the superposed deformation. We have, of course,

$$W_1 = w_1|_{i_1,i_2=I_1,I_2}, \qquad W_2 = w_2|_{i_1,i_2=I_1,I_2}. \tag{2.18}$$

We shall also use the notation W_{11}, W_{22}, $W_{12} = W_{21}$ to denote the values of $\partial^2 W/\partial i_1^2$, $\partial^2 W/\partial i_2^2$ and $\partial^2 W/\partial i_1 \partial i_2$ respectively at $i_1 = I_1$, $i_2 = I_2$. Then

$$\bar{w}_1 = W_{11}\bar{i}_1 + W_{12}\bar{i}_2, \qquad \bar{w}_2 = W_{21}\bar{i}_1 + W_{22}\bar{i}_2. \tag{2.19}$$

With (2.13), this yields

$$\begin{aligned} \bar{w}_1 &= 2\iota\omega S \sum_{A=1}^{3} \{W_{11} + (I_1 - \lambda_A^2)W_{12}\}\lambda_A^2 U_A l_A \exp[\iota\omega(Sl_i X_i - t)] \\ \bar{w}_2 &= 2\iota\omega S \sum_{A=1}^{3} \{W_{21} + (I_1 - \lambda_A^2)W_{22}\}\lambda_A^2 U_A l_A \exp[\iota\omega(Sl_i X_i - t)]. \end{aligned} \tag{2.20}$$

It follows from (2.16), (2.15), (2.18) and (2.11) that

$$\begin{aligned} T_{AA} &= 2[(W_1 + I_1 W_2)\lambda_A^2 - W_2\lambda_A^4] \\ T_{AB} &= 0 \qquad (A \neq B) \end{aligned} \tag{2.21}$$

and

$$\bar{t}_{AB} = \bar{T}_{AB} \exp[\iota\omega(Sl_i X_i - t)], \tag{2.22}$$

where

$$\begin{aligned} \bar{T}_{AA} = 4\iota\omega S\{&[W_1 + (I_1 - 2\lambda_A^2)W_2]\lambda_A^2 U_A l_A + \sum_{B=1}^{3} \lambda_A^2\lambda_B^2[W_{11} + (2I_1 - \lambda_A^2 - \lambda_B^2)W_{12} \\ &+ (I_1 - \lambda_A^2)(I_1 - {}_B^2)W_{22} + W_2]U_B l_B\} \end{aligned} \tag{2.23}$$

and

$$\bar{T}_{AB} = 2\iota\omega S[W_1 + (I_1 - \lambda_A^2 - \lambda_B^2)W_2](\lambda_A^2 l_A U_B + \lambda_B^2 l_B U_A) \qquad (A \neq B).$$

In the absence of body forces, the Cauchy stress σ_{ij}, given by (2.14) and (2.15), must satisfy the equations of motion

$$\frac{\partial \sigma_{ij}}{\partial x_j} = \frac{\partial t_{ij}}{\partial x_j} - \frac{\partial p}{\partial x_i} = \rho \ddot{u}_i, \tag{2.24}$$

where ρ is the density of the material. Introducing (2.16), bearing in mind that T_{ij} is constant and taking P = const., we obtain

$$\frac{\partial T_{ij}}{\partial X_j} - \frac{\partial P}{\partial X_i} = 0$$

and

$$\frac{\partial \bar{t}_{ij}}{\partial X_j} - \frac{\partial \bar{p}}{\partial X_i} = \rho \ddot{u}_i, \tag{2.25}$$

the first of these equations being identically satisfied. In the second of equations (2.25), we introduce

$$\bar{p} = \bar{P} \exp \iota\omega(Sl_i X_i - t), \tag{2.26}$$

where $\bar{P}$ is a complex constant. Then, using (2.2), (2.22) and (2.26), we obtain

$$\iota S(l_j \bar{T}_{ij} - \bar{P} l_i) = -\rho\omega U_i. \tag{2.27}$$

3. PROPAGATION IN A PRINCIPAL PLANE

We shall now consider that the wave is propagated in the x_1x_2-plane, so that $l_3 = 0$, and that the displacement vector is linearly polarized in this plane, so that $U_3 = 0$. Then, from (2.27), we obtain

$$\iota S(l_1 \bar{T}_{11} + l_2 \bar{T}_{12} - \bar{P} l_1) = -\rho\omega U_1$$

and

$$\iota S(l_1 \bar{T}_{12} + l_2 \bar{T}_{22} - \bar{P} l_2) = -\rho\omega U_2. \tag{3.1}$$

The third equation of motion is automatically satisfied. Eliminating $\bar{P}$ from equations (3.1), we obtain

$$\iota S[l_1 l_2(\bar{T}_{11} - \bar{T}_{22}) + (l_2^2 - l_1^2)\bar{T}_{12}] = -\rho\omega(U_1 l_2 - U_2 l_1). \tag{3.2}$$

The incompressibility condition given by the last of equations (2.13) yields

$$U_1 l_1 + U_2 l_2 = 0. \tag{3.3}$$

We note that this relation implies that the wave must be polarized in a direction perpendicular to the direction of propagation.

Then, from (3.2), (3.3) and (2.23), we obtain

$$2S^2(W_1 + \lambda_3^2 W_2)[\lambda_1^2 l_1^2 + \lambda_2^2 l_2^2 + l_1^2 l_2^2(\lambda_1 - \lambda_2)^2 A]U_1 = \rho U_1. \tag{3.4}$$

where

$$A = \frac{2(\lambda_1 + \lambda_2)^2}{W_1 + \lambda_3^2 W_2}(W_{11} + 2\lambda_3^2 W_{12} + \lambda_3^4 W_{22}). \tag{3.5}$$

We note that provided U_1 and U_2 are non-zero, the condition

$$2S^2(W_1 + \lambda_3^2 W_2)[\lambda_1^2 l_1^2 + \lambda_2^2 l_2^2 + l_1^2 l_2^2(\lambda_1 - \lambda_2)^2 A] = \rho \tag{3.6}$$

must be satisfied.

4. MATERIAL INSTABILITY CRITERIA

It has been pointed out by Hayes and Rivlin [2] that for the material to be stable for given values of λ_1, λ_2, λ_3, it is necessary that the slowness associated with propagation in the deformed material of a plane linearly polarized sinusoidal wave of infinitesimal amplitude be real for all directions of propagation and for all possible directions of polarization. Therefore, for the material to be stable the condition $S^2 > 0$ must be satisfied for all possible values of l_1 and l_2, i.e., for

$$0 \leq l_1^2 \leq 1, \qquad 0 \leq l_2^2 \leq 1, \qquad l_1^2 + l_2^2 = 1. \tag{4.1}$$

We note also the condition that the slownesses be real for waves propagated in the direction (0, 1, 0) and linearly polarized in the x_1-direction. From (3.6) this condition is (cf. [3])

$$W_1 + \lambda_3^2 W_2 > 0. \tag{4.2}$$

That this is a necessary condition for material stability also results from the following consideration. The shear modulus must be positive for a small static shear in the x_1-direction, with the x_1x_2-plane, say, as the plane of shear superimposed on an underlying pure homogeneous deformation.

It follows from (3.6) and (4.2) that a necessary condition for material stability is

$$\Phi = \lambda_1^2 l_1^2 + \lambda_2^2 l_2^2 + l_1^2 l_2^2 (\lambda_1 - \lambda_2)^2 A > 0 \tag{4.3}$$

for all l_1, l_2 satisfying (4.1). We note that if $\lambda_1 = \lambda_2$, then this condition is satisfied for all A. If $\lambda_1 \neq \lambda_2$ and $A = -(\lambda_1 + \lambda_2)^2/(\lambda_1 - \lambda_2)^2$, we have

$$\Phi = (\lambda_1 l_1^2 - \lambda_2 l_2^2)^2. \tag{4.4}$$

Φ is then non-negative for all l_1, l_2 and becomes zero when

$$l_1^2/l_2^2 = \lambda_2/\lambda_1. \tag{4.5}$$

It is evident from (4.3) that for specified values of l_1 and l_2, Φ increases as A increases and decreases as A decreases. Thus, the condition that $\Phi > 0$ for all l_1, l_2 satisfying (4.1) is satisfied for $A > -(\lambda_1 + \lambda_2)^2/(\lambda_1 - \lambda_2)^2$. However, for

$$A < -(\lambda_1 + \lambda_2)^2/(\lambda_1 - \lambda_2)^2, \tag{4.6}$$

this condition is violated for l_1 and l_2 satisfying (4.5) and hence the material is inherently unstable. Introducing the expression (3.5) for A, the condition (4.6) becomes

$$B_3 = \frac{W_{11} + 2\lambda_3^2 W_{12} + \lambda_3^4 W_{22}}{W_1 + \lambda_3^2 W_2} < -\tfrac{1}{2}(\lambda_1 - \lambda_2)^{-2}. \tag{4.7}$$

Two further analogous conditions are obtained by considering the propagation of waves in the 23 and 31 planes. They are

$$B_1 = \frac{W_{11} + 2\lambda_1^2 W_{12} + \lambda_1^4 W_{22}}{W_1 + \lambda_1^2 W_2} < -\tfrac{1}{2}(\lambda_2 - \lambda_3)^{-2}$$

and

$$B_2 = \frac{W_{11} + 2\lambda_2^2 W_{12} + \lambda_2^4 W_{22}}{W_1 + \lambda_2^2 W_2} < -\tfrac{1}{2}(\lambda_3 - \lambda_1)^{-2}. \tag{4.8}$$

If any of these conditions is satisfied the material is unstable. If none of them is satisfied and $W_1 + \lambda_A^2 W_2 > 0$ $(A = 1, 2, 3)$, the material is stable with respect to waves propagated in the principal planes. We note that this is always the case for a Mooney–Rivlin material, for which $W_1 > 0$ and $W_2 > 0$. For such a material, $W_{11} = W_{12} = W_{22} = 0$. For many vulcanized rubbers it has been shown [4], [5] that W may, with reasonably good approximation be written in the form

$$W = C(I_1-3)+f(I_2-3), \tag{4.9}$$

where C is a constant and $W_2(=\partial f/\partial I_2)$ is a decreasing function of I_2-3, so that $W_{22} < 0$. The conditions (4.7) and (4.8) then place restrictions on the values which can be taken by W_{22}.

Finally, we note that if W_1 and W_2 and the quantities B_1, B_2 and B_3 defined in (4.7) and (4.8) are known, W_{11}, W_{12} and W_{22} are uniquely determined by the relations

$$\begin{aligned} W_{11} &= -\frac{1}{\Delta}[\lambda_2^2\lambda_3^2(\lambda_2^2-\lambda_3^2)A_1+\lambda_3^2\lambda_1^2(\lambda_3^2-\lambda_1^2)A_2+\lambda_1^2\lambda_2^2(\lambda_1^2-\lambda_2^2)A_3], \\ W_{12} &= \frac{1}{2\Delta}[(\lambda_2^4-\lambda_3^4)A_1+(\lambda_3^4-\lambda_1^4)A_2+(\lambda_1^4-\lambda_2^4)A_3], \\ W_{22} &= -\frac{1}{\Delta}[(\lambda_2^2-\lambda_3^2)A_1+(\lambda_3^2-\lambda_1^2)A_2+(\lambda_1^2-\lambda_2^2)A_3], \end{aligned} \tag{4.10}$$

where

$$\Delta = (\lambda_2^2-\lambda_3^2)(\lambda_3^2-\lambda_1^2)(\lambda_1^2-\lambda_2^2).$$

and

$$A_P = B_P(W_1+\lambda_P^2 W_2) \qquad (P = 1,2,3). \tag{4.11}$$

Thus measurement of the velocity of a wave propagated and polarized in each of the principal planes determines W_{11}, W_{12} and W_{22}. It can also be seen [3] that the measurement of the velocity of a wave propagated in each of two principal directions and polarized in the third principal direction determines W_1 and W_2.

Measurement of these five wave speeds is adequate for the determination of the wave speed in an arbitrary direction, not necessarily in a principal plane, since this wave speed is also determined by W_1, W_2, W_{11}, W_{12}, W_{22}.

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Абстракт—Определяются необходимые условия для устойчивости несжимаемых упругих материалов. Они получаются путем рассмотрения скоростей плоских волн малой амплитуды наложенных на конечно деформированное состояние материала.